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MASS CALIBRATION REVISITED

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LABORATOIRE NATIONAL DE MÉTROLOGIE ET D'ESSAIS

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EMUE Examples of Measurement Uncertainty Evaluation

OBJECTIVES

To compare GUF (GUM Uncertainty Framework), MCM (Monte Carlo Method) and Bayesian approach

- **mass calibration JCGM 101:2008 :** no measurements but a best estimate and its associated uncertainty

- **mass calibration revisited :** N measurements collected during a calibration process

MASS CALIBRATION EXAMPLE

Mass calibration example according to JCGM 101:2008

MASS CALIBRATION EXAMPLE: CALIBRATION OF A WEIGHT W AGAINST A REFERENCE WEIGHT R

Measurement model

$$\delta_{\mathrm{m}} = m_{\mathrm{W,c}} - m_{\mathrm{nom}} = \left(m_{\mathrm{R,c}} + \delta m_{\mathrm{R,c}}\right)\left(1 + \left(\rho_{\mathrm{a}} - \rho_{\mathrm{a_0}}\right)\left(\frac{1}{\rho_{\mathrm{W}}} - \frac{1}{\rho_{\mathrm{R}}}\right)\right) - m_{\mathrm{nom}}$$

$$\boxed{\delta_{\mathrm{m}} = C\left(m_{\mathrm{R,c}} + \delta m_{\mathrm{R,c}}\right) - m_{\mathrm{nom}}}$$

- δ_{m} : measurand: deviation of $m_{\mathrm{W,c}}$ from the nominal mass
- $\delta m_{\mathrm{R,c}}$: deviation from reference conventional mass known either from measurements or a best estimate and its associated uncertainty
- $m_{\mathrm{R,c}}$: conventional reference mass
- $C = 1 + \left(\rho_{\mathrm{a}} - \rho_{\mathrm{a_0}}\right)\left(\frac{1}{\rho_{\mathrm{W}}} - \frac{1}{\rho_{\mathrm{R}}}\right)$: correction
- m_{nom} : nominal mass
- $\rho_{\mathrm{a_0}} = 1.2\ \mathrm{kg/m^3}$, ρ_a : mass density of air
- ρ_{W}, ρ_{R}: mass density of resp. the weight W and reference weight R

FROM THE MEASUREMENT MODEL TO THE STATISTICAL MODEL

Notations and modelling in the GUM framework

measurement model of the form $Y = f(X)$

Y the measurand, X the input quantities, f the measurement model

Notations and statistical modelling

η the measurand,
ξ the input quantity usually associated with measurements,
θ the other input quantities,

$$\xi = g(\eta, \theta) + \varepsilon$$

Assumptions: $\varepsilon \sim N(0, \sigma^2)$,
one-to-one correspondance between ξ and η

STATISTICAL MODEL

Statistical model $\delta m_{\mathrm{R,c}} = \frac{\delta_{\mathrm{m}}}{C} - \frac{C m_{\mathrm{R,c}} - m_{\mathrm{nom}}}{C} + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2)$

Correspondance with the model $\xi = \frac{\eta - D(\theta)}{C(\theta)} + \varepsilon$

$$\xi = \delta m_{\mathrm{R,c}}$$
$$\eta = \delta_{\mathrm{m}}$$
$$\theta = (\rho_{\mathrm{a}}, \rho_{\mathrm{W}}, \rho_{\mathrm{R}}, m_{\mathrm{R,c}})$$
$$C(\theta) = 1 + (\rho_{\mathrm{a}} - \rho_{\mathrm{a}_0})\left(\frac{1}{\rho_{\mathrm{W}}} - \frac{1}{\rho_{\mathrm{R}}}\right)$$
$$D(\theta) = C(\theta) m_{\mathrm{R,c}} - m_{\mathrm{nom}}$$

MASS CALIBRATION EXAMPLE IN JCGM 101

Posterior distribution under non informative prior $\pi(\delta_m) \propto 1$

$$\pi(\delta_m|d,\theta) \propto \frac{1}{(s^2)^{\frac{1}{2}}} \exp - \frac{1}{2C^2 s^2}(\mu - m)^2$$

Gaussian distribution

where $m = Cd + \delta$ and $\theta = (C, \delta)$, d is a best estimate with assoc. uncertainty $u(d) = s$ $(\sigma^2 = s^2)$

Propagation of uncertainty:

marginal posterior distribution

$$\boxed{\pi(\delta_m|d) \propto \int \frac{1}{(s^2)^{\frac{1}{2}}} \exp - \frac{1}{2C^2 s^2}(\mu - m)^2 d\theta}$$

COMPARISON OF RESULTS FOR THE MASS CALIBRATION EXAMPLE IN JCGM 101

Results

JCGM 101

Table 6 — Results of the calculation stage for the mass calibration model (24) (9.3.2.1, 9.3.2.6)

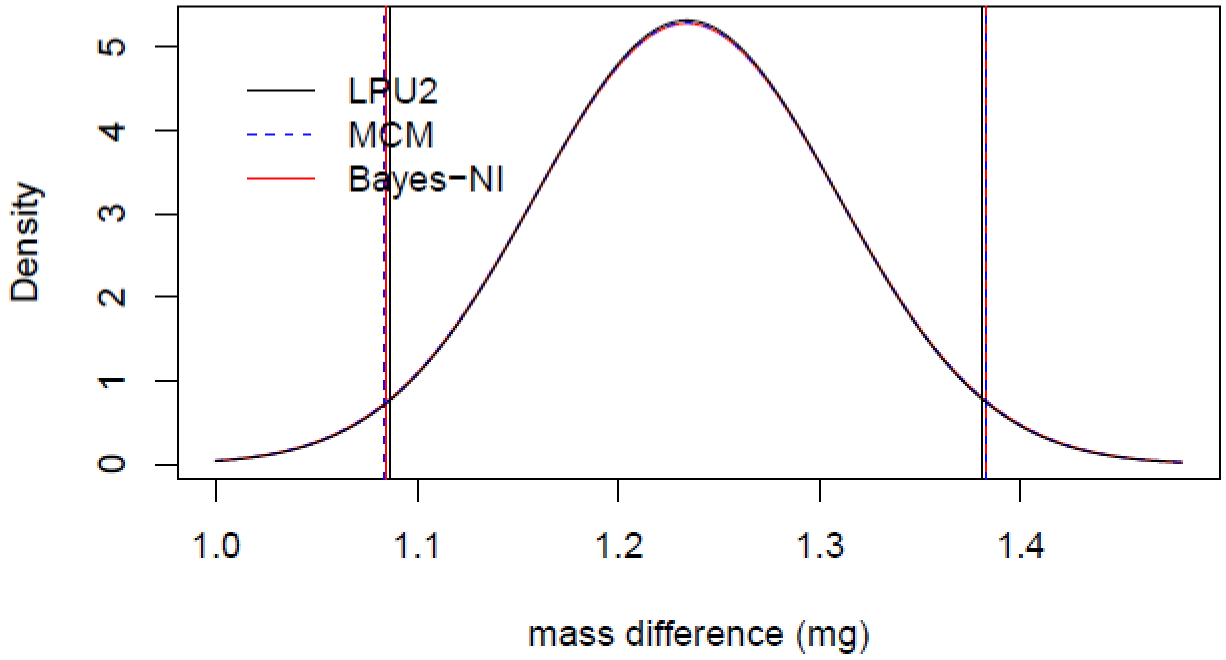
Method	$\widehat{\delta m}$ /mg	$u(\widehat{\delta m})$ /mg	Shortest 95 % coverage interval /mg	d_{low} /mg	d_{high} /mg	GUF validated ($\delta = 0.005$)?
GUF_1	1.234 0	0.053 9	[1.128 5, 1.339 5]	0.045 1	0.043 0	No
MCM	1.234 1	0.075 4	[1.083 4, 1.382 5]			
GUF_2	1.234 0	0.075 0	[1.087 0, 1.381 0]	0.003 6	0.001 5	Yes

Bayes-NI $\widehat{\delta_m} = 1.233\ 95$ mg, $u(\widehat{\delta_m}) = 0.075\ 37$ mg

$d_{low} = 1.084\ 55$ mg, $d_{high} = 1.382\ 97$ mg

Similar results for LPU2, MCM and the Bayesian approach

COMPARISON OF RESULTS FOR THE MASS CALIBRATION EXAMPLE IN JCGM 101



COMPARISON BETWEEN GUMS1 ASSUMPTIONS AND STATISTICAL MODELLING ASSUMPTIONS

Mass calibration example according to JCGM 101

- σ^2 known ($\sigma^2 = s^2$)
- a best estimate d and its associated uncertainty $u(d)$

Warning In mass calibration example from JCGM 101, type B uncertainty evaluation (no measurements)

Mass calibration example in the general case

- σ^2 unknown (to be estimated)
- N measurements $d_1, \ldots, d_N$

Warning In the following, adaptation of the mass calibration example to the case where measurements are available

MASS CALIBRATION EXAMPLE

Mass calibration example in the general case

ADAPTATION OF THE MASS CALIBRATION EXAMPLE IN THE GENERAL CASE

Posterior distribution for the conjugate case

Conjugate prior: $\delta_m \sim N\left(\eta_0, \frac{\sigma^2}{\kappa_0}\right), \sigma^2 \sim Inv - chi2(\nu_0, s_0^2)$

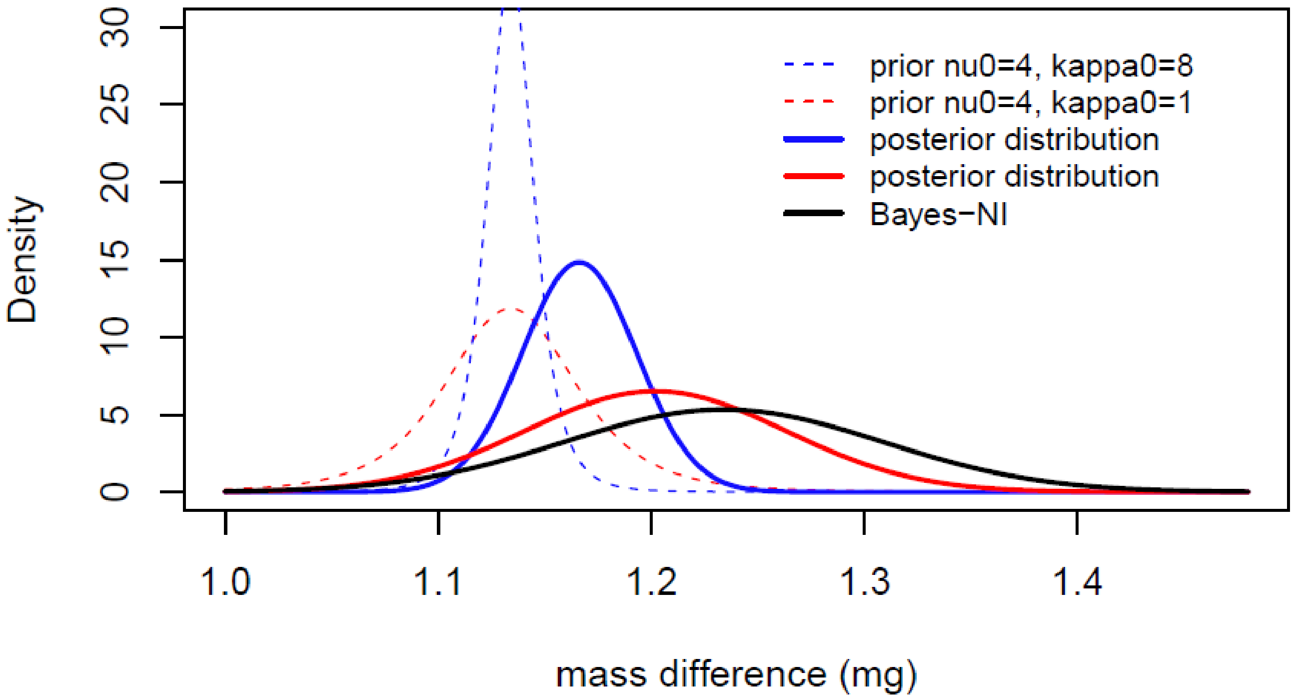
Posterior: $$\boxed{\pi(\delta m|d) \propto \int_\theta t_{N+\nu_0}\left(\delta m; \eta_N(\theta), \frac{\hat{\tau}_N^2}{\kappa_N(\theta)}\right)\pi(\theta)d\theta}$$

arising from the integration over σ^2

$$\eta_N = \frac{1}{\kappa_N(\theta)}\left(\frac{N}{C(\theta)^2}m_N + \kappa_0\eta_0\right)$$

$$\hat{\tau}_N^2 = \frac{\frac{\kappa_0 N}{\kappa_N C(\theta)^2}(\eta_N-\eta_0)^2+(N-1)s_N^2+\nu_0 s_0^2}{N+\nu_0} \qquad \kappa_N(\theta) = \frac{N}{C(\theta)^2} + \kappa_0$$

EFFECT OF PRIOR PARAMETERS ON THE POSTERIOR DISTRIBUTION IN THE CONJUGATE CASE



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EXAMPLES OF NON CONJUGATE PRIOR DISTRIBUTIONS

Unbounded prior distributions

- Gaussian

$$\pi(\eta|\sigma^2) \propto \frac{1}{(\sigma_0^2)^{\frac{1}{2}}} \exp - \frac{1}{2\sigma_0^2}(\eta - \mu_0)^2$$

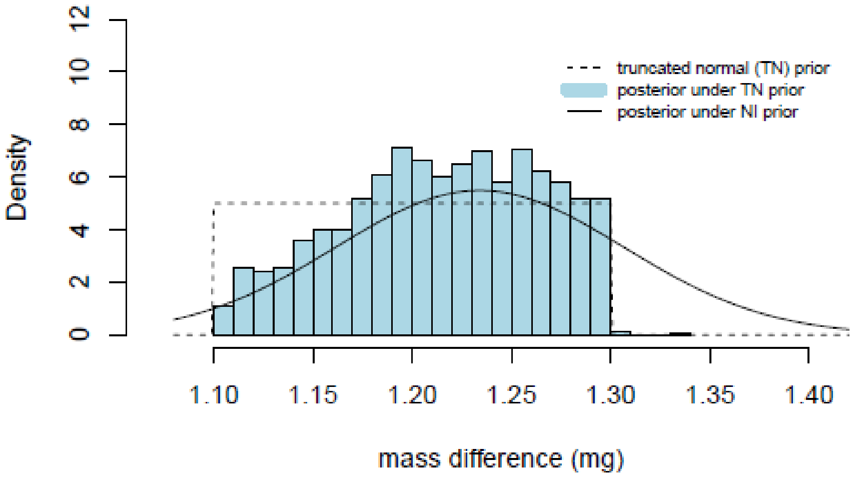
Bounded prior distributions on [a,b]

- Truncated Gaussian distribution
- Uniform distribution

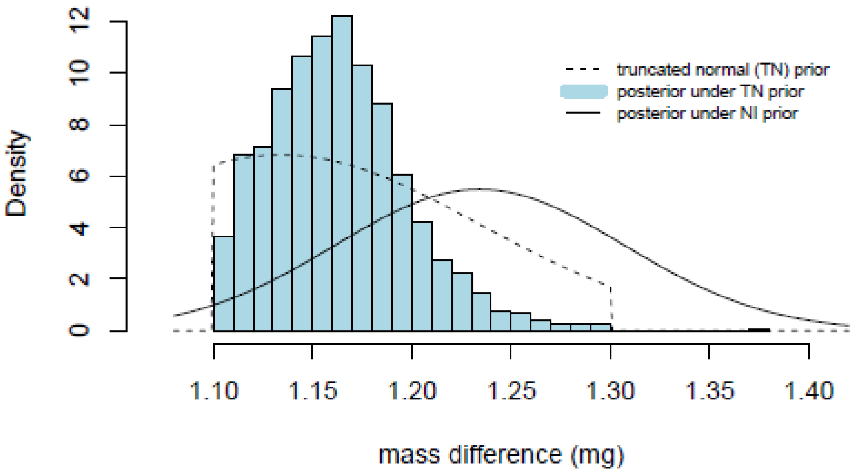
EFFECT OF NON CONJUGATE PRIOR ON THE POSTERIOR DISTRIBUTION

Bounded prior → bounded posterior !

Uniform prior



Truncated normal prior



CONCLUSION

- Similar results for LPU, MCM and Bayes (with noninformative prior) can be reached for the mass example when sigma is known, but

- Bayes allows to incorporate prior knowledge about the measurand which can improve that result (i.e. sharpen the posterior) significantly,

- Care needs to be taken when assigning an informative prior as that can be quite influential (see observed significant sensitivities),

- In general: Bayes provides a flexible tool for statistical modeling, achieves added value through prior info, at some computational price

- Demonstrator of this application under development at LNE, available in 2020

REFERENCES

BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP, and OIML. *Supplement 1 to the 'Guide to the Expression of Uncertainty in Measurement' – Propagation of distributions using a Monte Carlo method, JCGM 101:2008*. BIPM, 2008.

C. Elster. Bayesian uncertainty analysis compared with the application of the GUM and its supplements. *Metrologia*, 51:S159–S166, 2014.

Thank you for your attention

STATISTICAL MODEL

General form of models considered

$$\xi = \frac{\eta - D(\theta)}{C(\theta)} + \varepsilon$$

$$\varepsilon \sim N(0, \sigma^2)$$

Similar to the problem of estimating the mean of a Gaussian distribution
2 cases are usually considered:

- $\boldsymbol{\sigma^2}$ known
- $\boldsymbol{\sigma^2}$ unknown (general case)

LIKELIHOOD & PRIOR

Case σ^2 unknown

$$l(d|\eta,\sigma^2,\theta) \propto \underbrace{\frac{1}{\left(\frac{\sigma^2}{N}\right)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2\frac{C^2\sigma^2}{N}}(\eta - m_N)^2\right\}}_{\text{Gaussian distribution}} \times \underbrace{\frac{1}{(\sigma^2)^{\frac{N-1}{2}}} \exp\left\{-\frac{(N-1)S_N^2}{2\sigma^2}\right\}}_{\text{Inverse Chi2 distribution}}$$

where $m_N = C\bar{d} + \delta$

$\bar{d}$ is the sample mean, s_N^2 is the sample variance, N is the sample mean

A <u>prior</u> conjugate with the likelihood is of the form

$$\pi(\eta|\sigma^2) \propto \frac{1}{\left(\frac{\sigma^2}{\kappa_0}\right)^{\frac{1}{2}}} \exp - \frac{1}{2\frac{\sigma^2}{\kappa_0}}(\eta - \mu_0)^2,\ \pi(\sigma^2) \propto Inv - Chi2(\nu_0, s_0^2)$$

LIKELIHOOD & PRIOR

Case σ^2 known ($\sigma^2 = s^2$)

$$l(d|\eta,\theta) \propto \frac{1}{\left(\frac{s^2}{N}\right)^{\frac{1}{2}}} \exp - \frac{1}{\frac{2C^2 s^2}{N}} (\eta - m_N)^2$$

where $m_N = C\bar{d} + \delta$,

Remark: d is a best estimate with assoc. uncertainty $u(d) = s$, set $N = 1$

A prior conjugate with the likelihood is of the form

$$\pi(\eta|\sigma^2) \propto \frac{1}{(\sigma_0^2)^{\frac{1}{2}}} \exp - \frac{1}{2\sigma_0^2} (\eta - \mu_0)^2$$

POSTERIOR DISTRIBUTION OF THE MEASURAND

Non informative prior distributions

Conjugate prior distributions

Non conjugate prior distributions

Choice of prior distribution $\pi(\eta, \sigma^2)$

likelihood

likelihood

$$l(d|\eta, \sigma^2, \theta)$$

Analytical formulas

MCMC simulations

$$\pi(\eta, \sigma^2, \theta|d)$$

Monte Carlo integration on θ

$$\pi(\eta|d) = \int\int l(d|\eta, \sigma^2, \theta)\pi(\eta, \sigma^2)\pi(\theta)d\sigma^2 d\theta$$