Physikalisch-Technische Bundesanstalt
Braunschweig und Berlin
Nationales Metrologieinstitut

Generative models as prior in Bayesian inverse problems

M. Marschall, G. Wübbeler, F. Schmähling and C. Elster

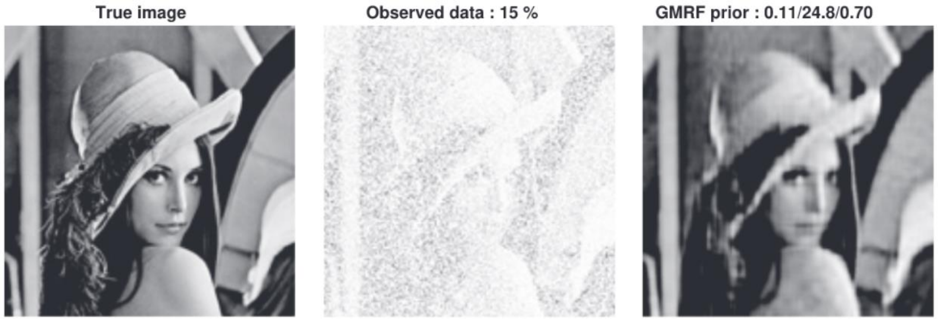
Motivation: inverse problem

$$y = f(x)$$

Given observation y, find unknown x.

Example: Image reconstruction

(inpainting, deblurring, denoising,
(non-) parametric regression, …)



Example solution: **Least-squares** approach

$$\min_{\mathrm{x}} ||y - f(x)||^2 + \mathcal{R}(\mathrm{x})$$

Q: How to choose the regularizer $\mathcal{R}(x)$?

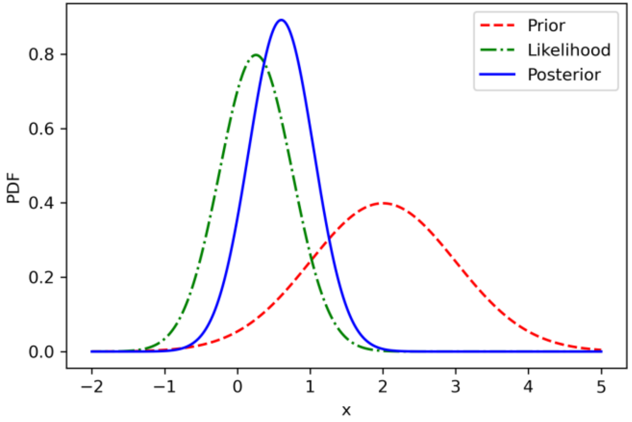
Motivation: Bayesian inverse problem

$$y|x,\sigma^2 \sim N(f(x),\sigma^2 I)$$
$$x \sim \pi(x)$$

Given observation y, variance σ^2 and prior $\pi(x)$, find posterior of unknown x.

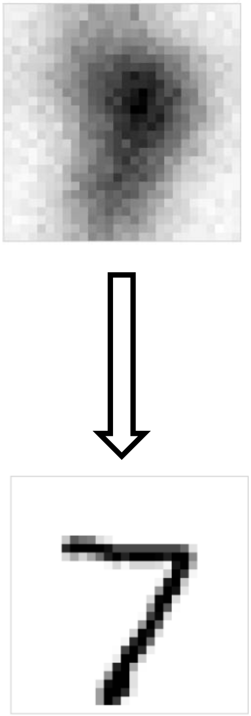
Bayes' Theorem

$$\pi(x \mid y) \propto L(y \mid x)\,\pi(x)$$



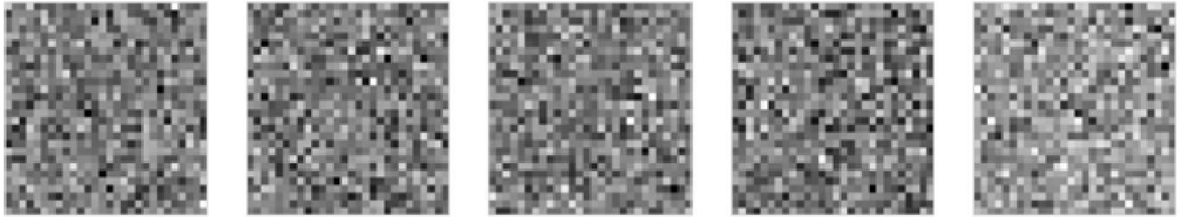
Q: How to choose the prior $\pi(x)$?

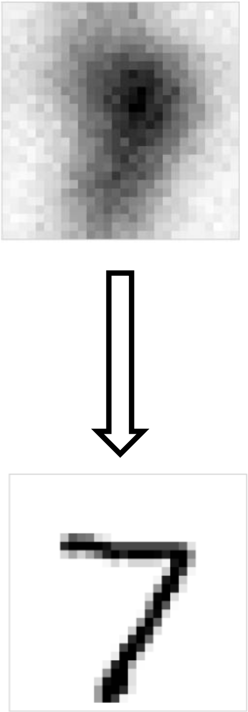
Choosing the prior - visualized



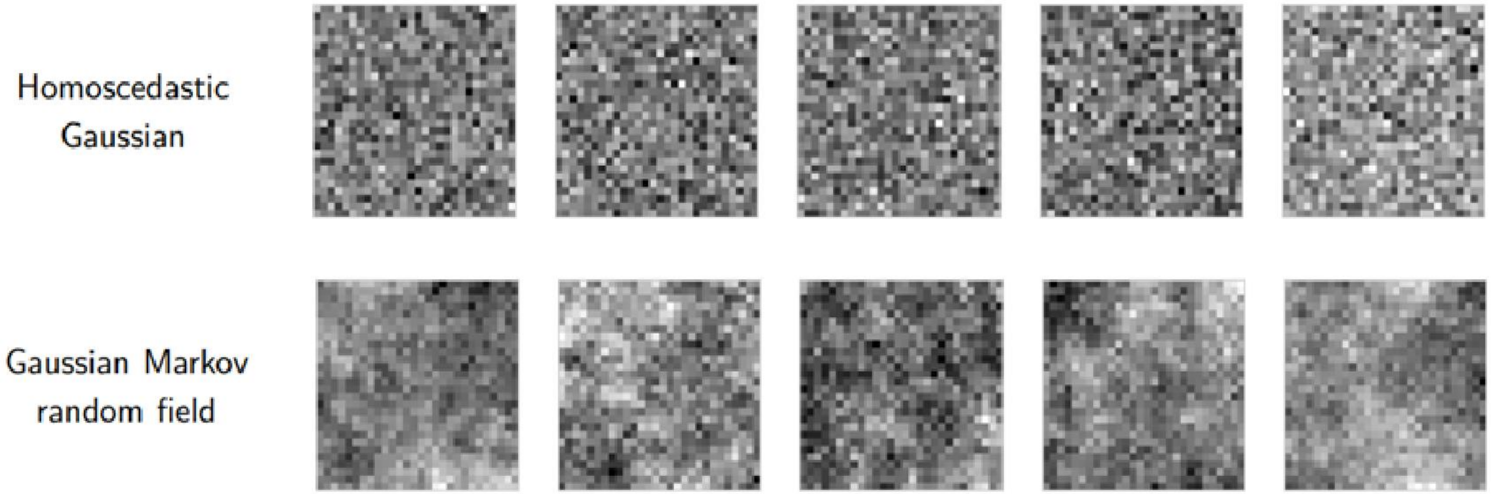
Choosing the prior - visualized

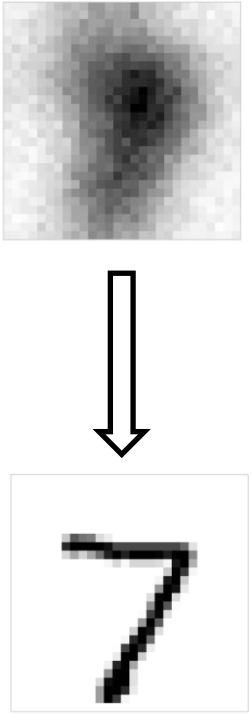
Homoscedastic Gaussian



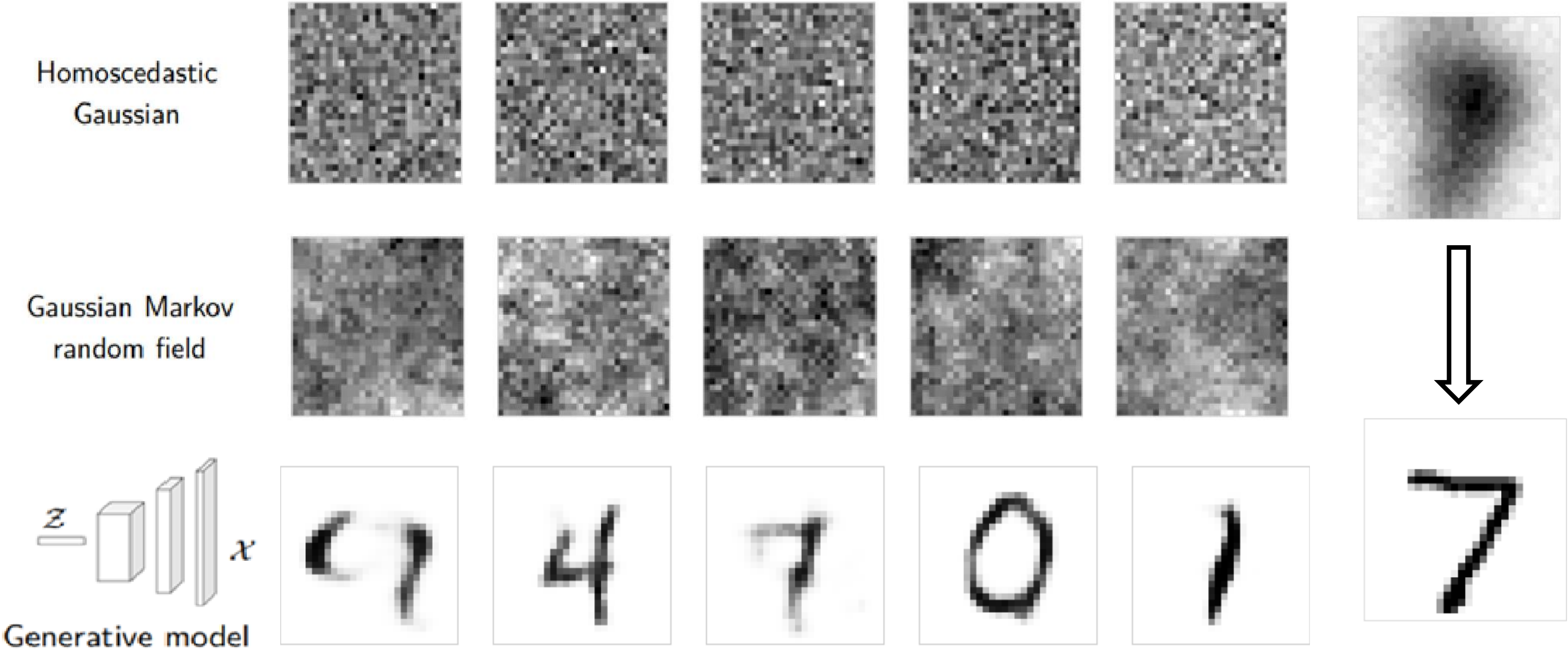


Choosing the prior - visualized

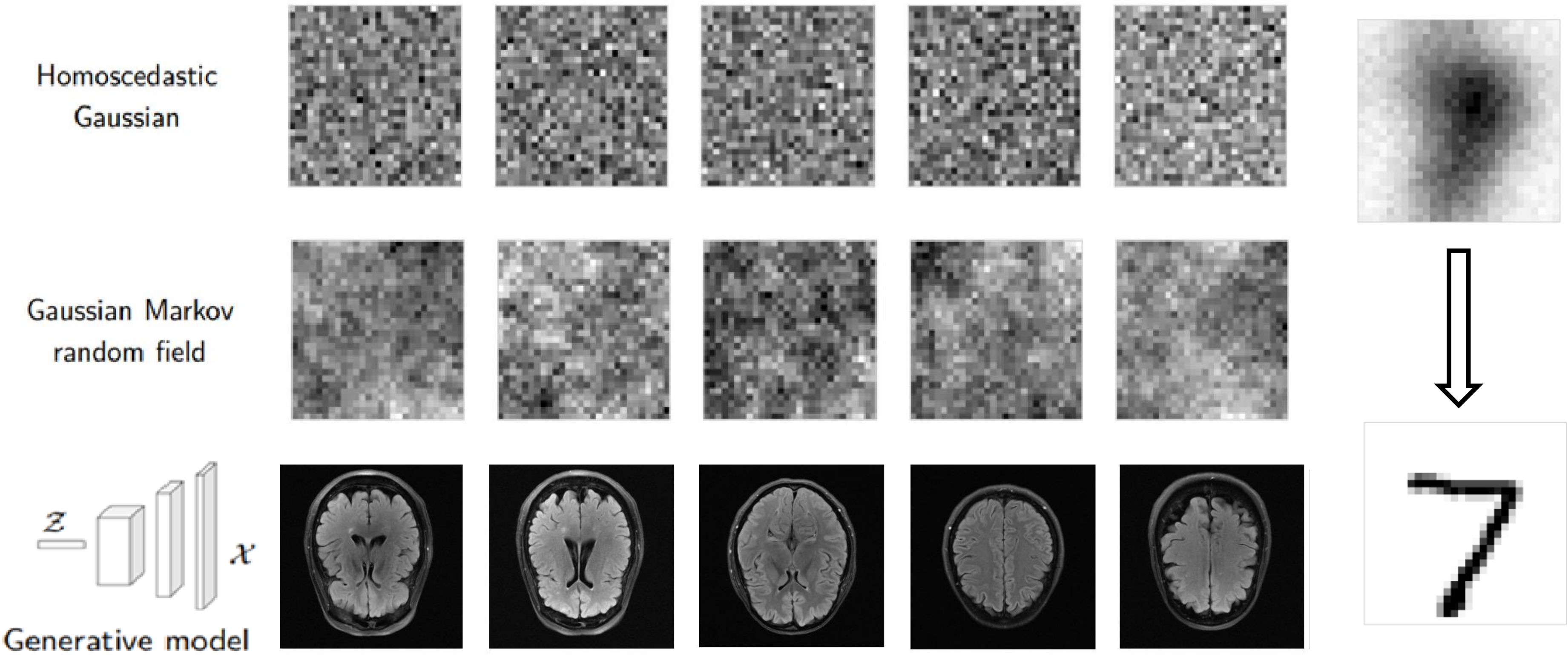




Choosing the prior - visualized

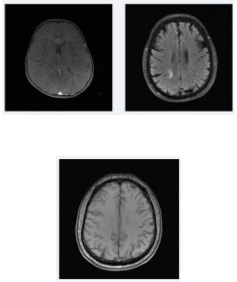


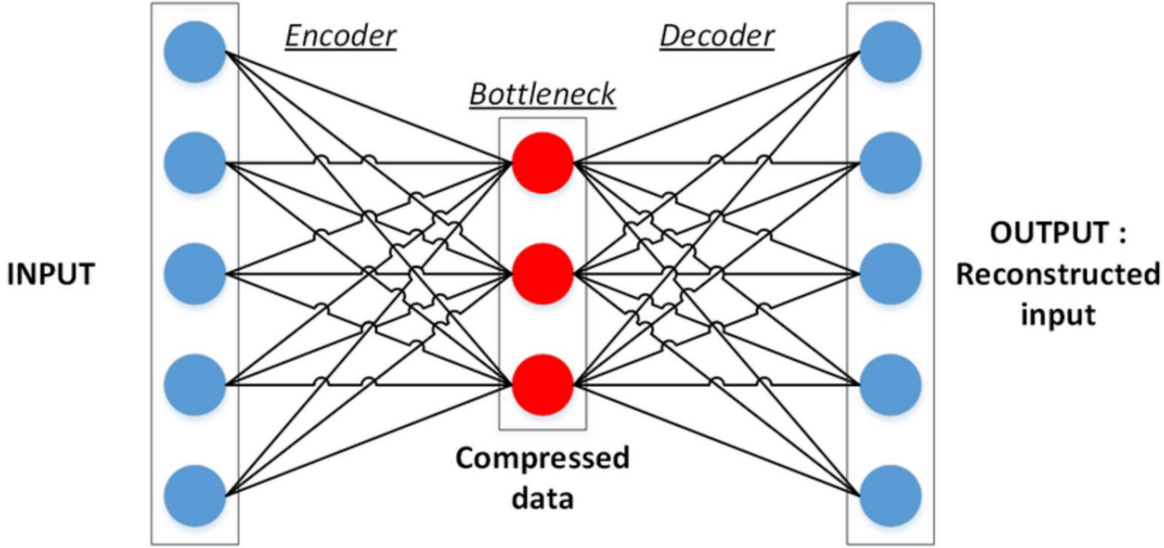
Choosing the prior - visualized

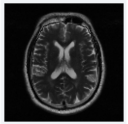


Generative models in machine learning

From latent space to variable space: AE / VAE / GAN / … / KL / PCE / … / PCA / …



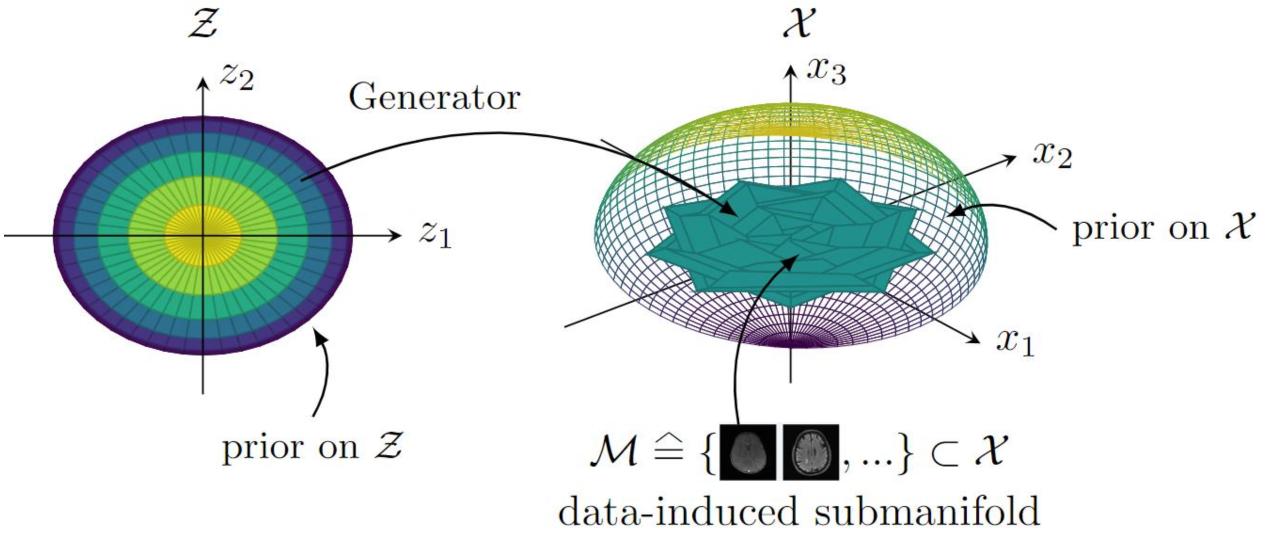




© miro.medium.com

Variational autoencoder: $z \,\middle|\, x \sim N\left(e_{\theta_1}(x), \Sigma_{\theta_2}(x)\right),\ x \,\middle|\, z \sim N\left(g_{\theta_3}(z), \Gamma_{\theta_4}(z)\right)$

Latent space vs. Variable space



Inference in latent space

$$y|z \sim N(Ag(z), \sigma^2 I)$$

- Low-dimensional but highly non-linear
- Lemma: Bayes estimators are inconsistent

Inference in variable space

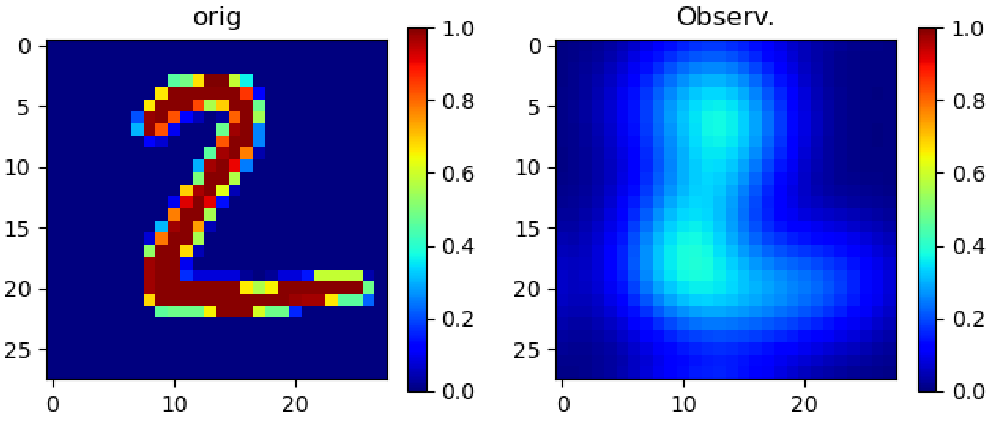
$$y|x \sim N(Ax, \sigma^2 I)$$

- High-dimensional but linear problem
- Lemma: Bayes estimators are consistent

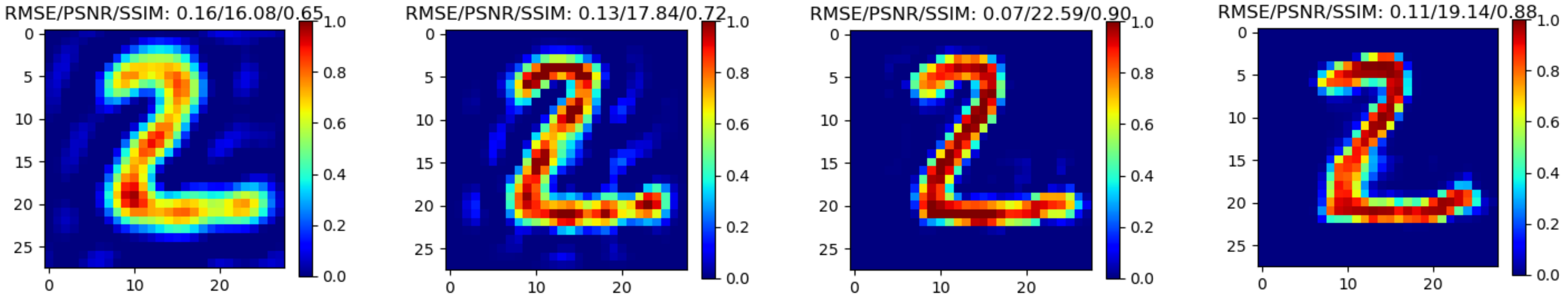
Holden M., Marcelo P. and Konstantinos C. Z. “Bayesian Imaging with Data-Driven Priors Encoded by Neural Networks." *SIAM Journal on Imaging Sciences* 15.2 (2022): 892-924.

MM, Wübbeler G., Schmähling F. and Elster C. “Generative models and Bayesian inversion using Laplace approximation,” **2022, https://arxiv.org/pdf/2203.07755.pdf**

Numerical results – comparison of inference methods

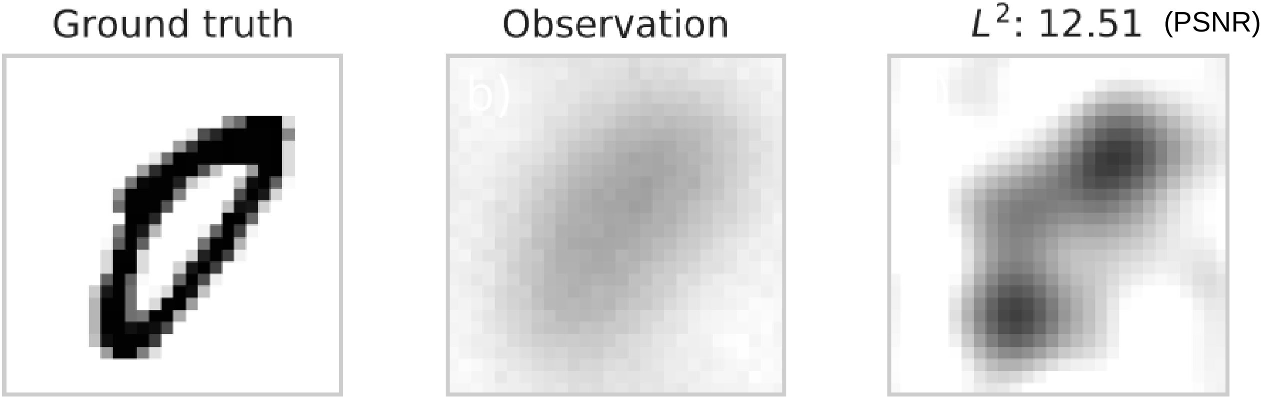


$$\sigma = 0.001$$

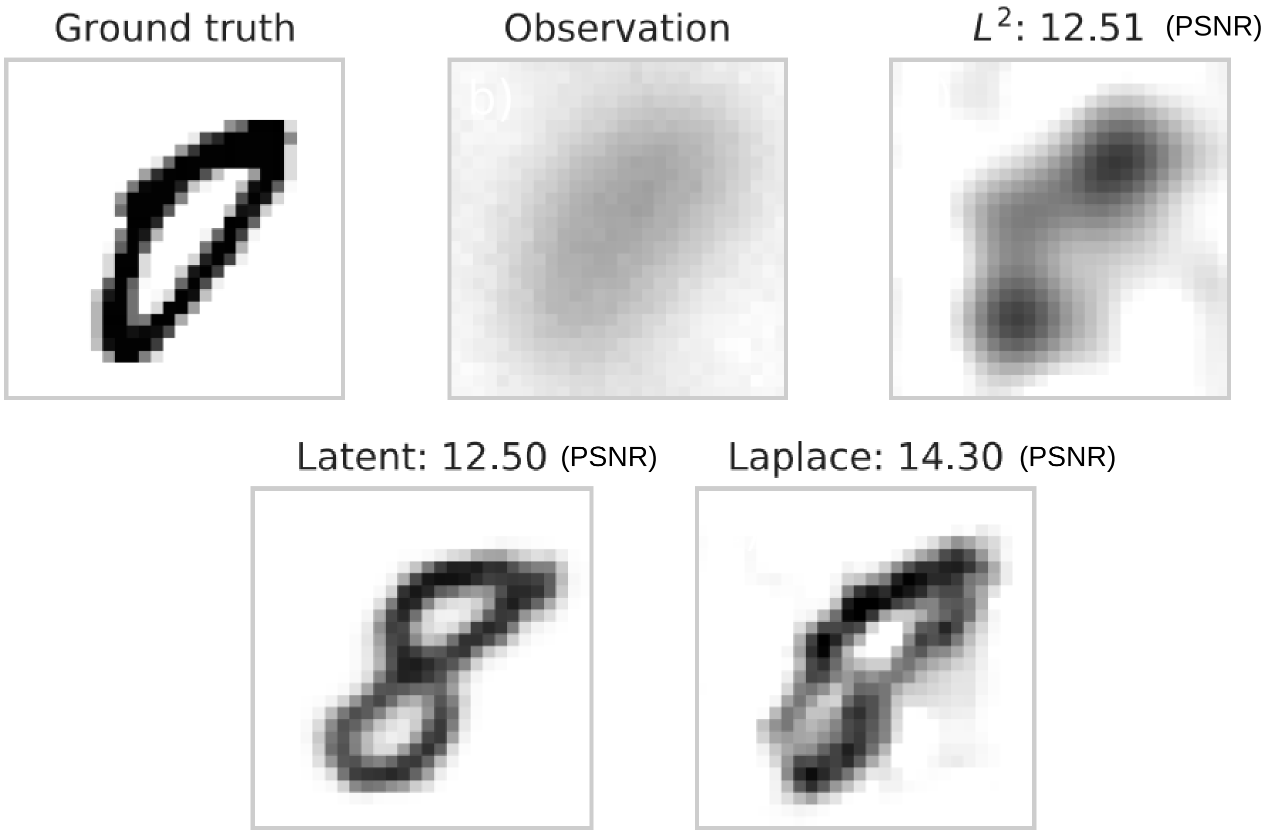


$N(0,\lambda I)$ | $N(mean, cov)$ | $Laplace$ | $Latent$

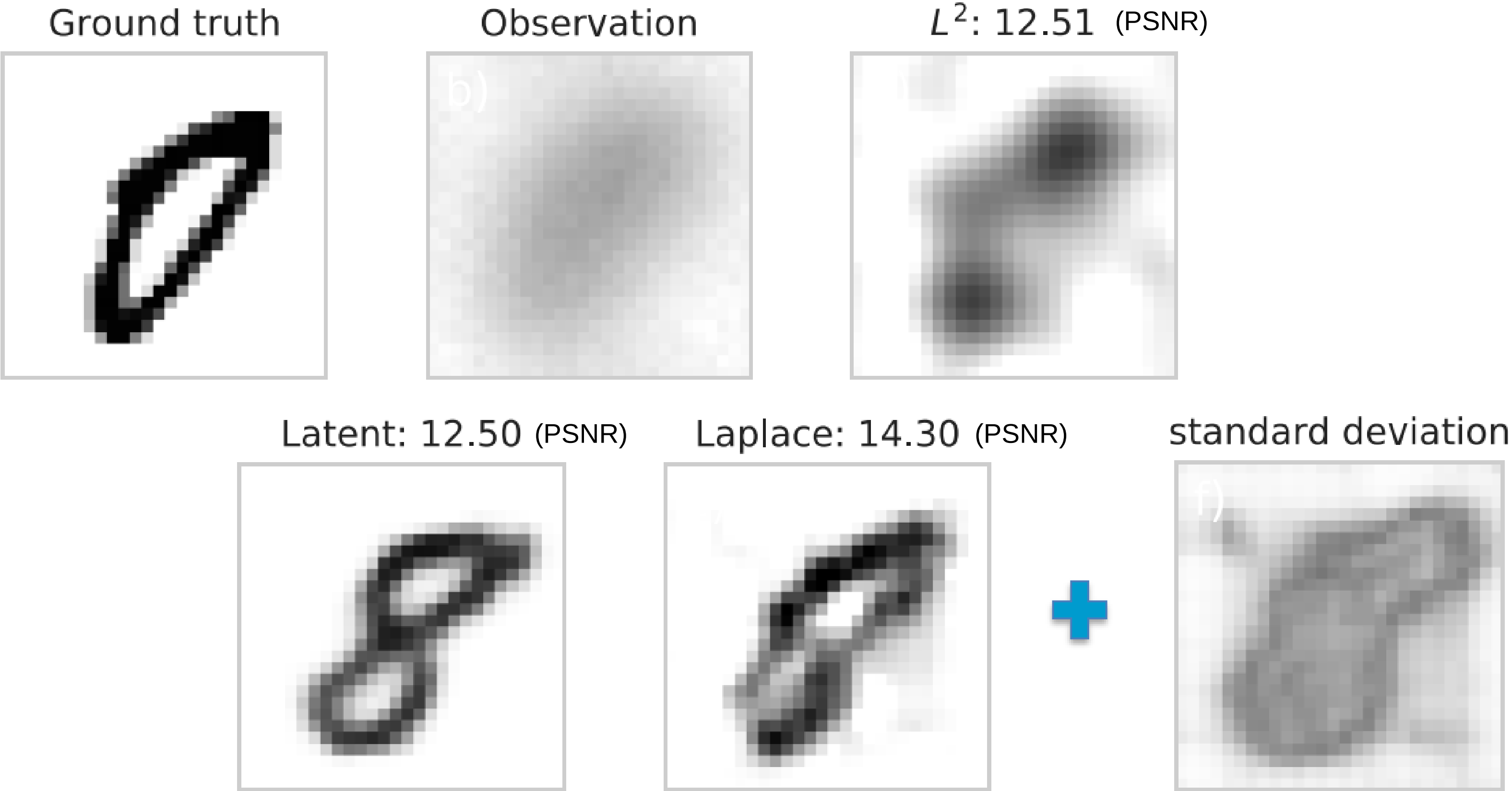
Numerical results – comparison of inference methods



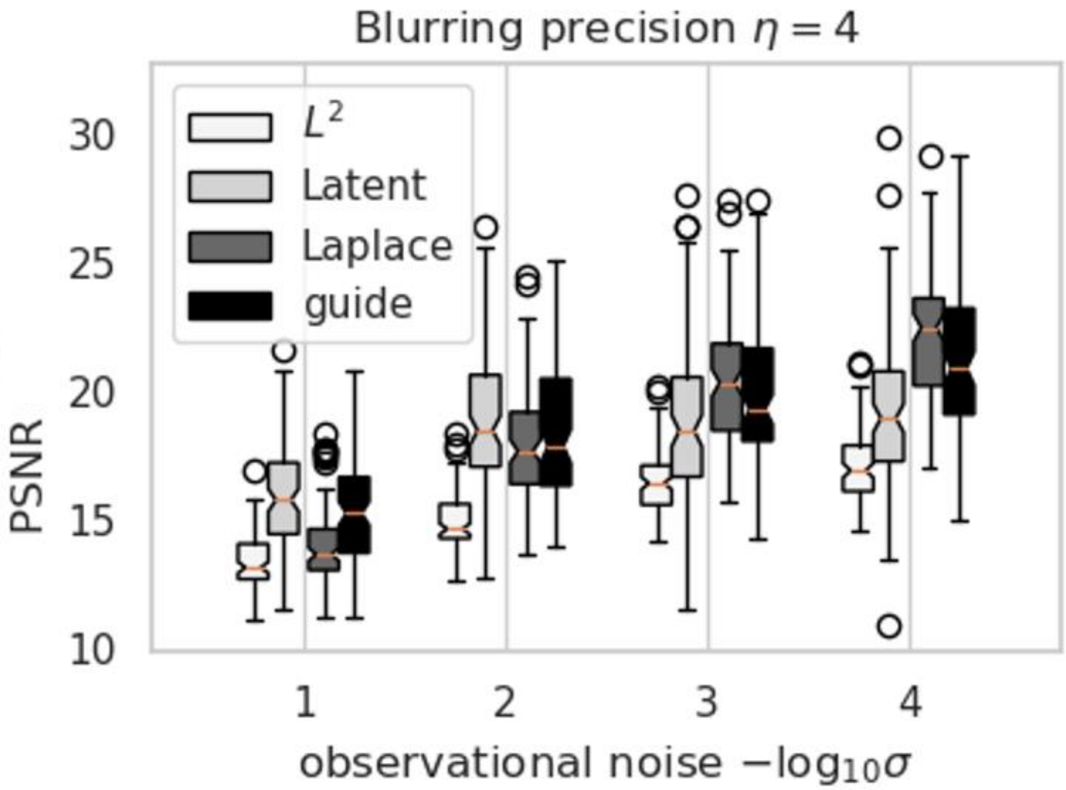
Numerical results – comparison of inference methods



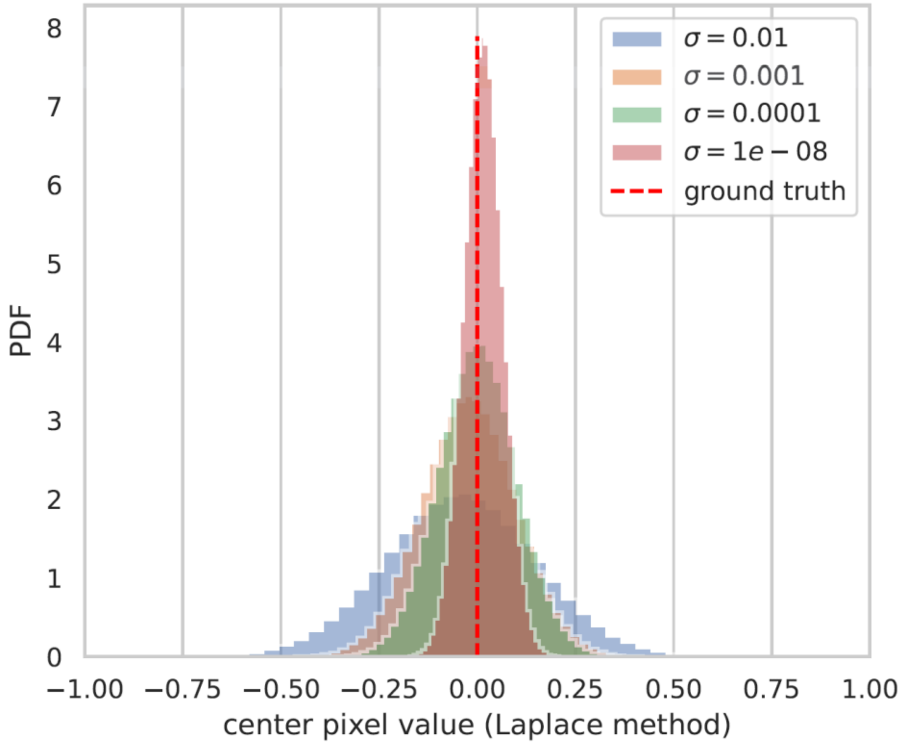
Numerical results – comparison of inference methods



Numerical results – consistency and guidance



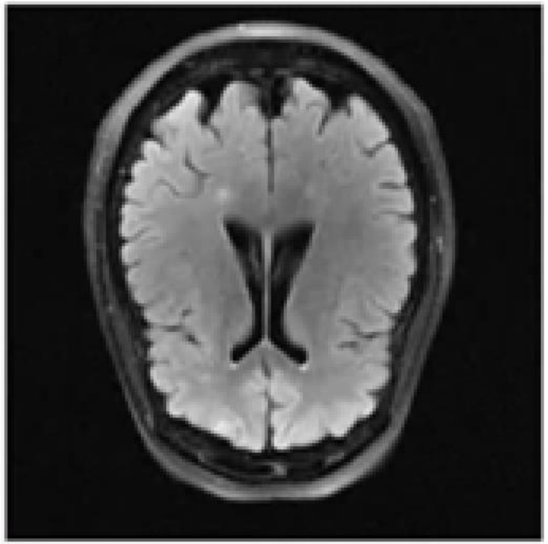
Numerical results - consistency



Application: qualitative MRI

Observation in Fourier domain
y

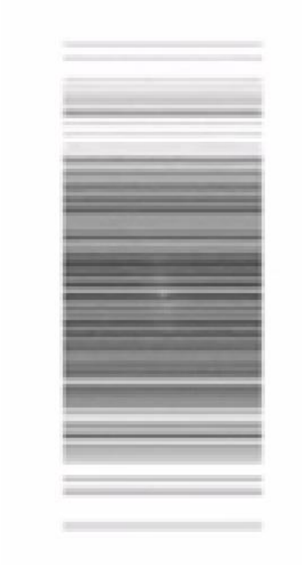
Reconstructed magnetziation
x



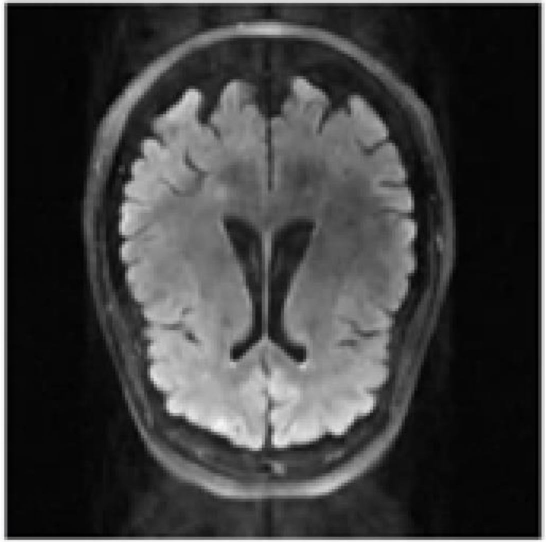
$$y|x,\sigma^2 \sim N(Ax, \sigma^2 I)$$

Application: qualitative MRI

Observation in Fourier domain
y



Reconstructed magnetziation
x



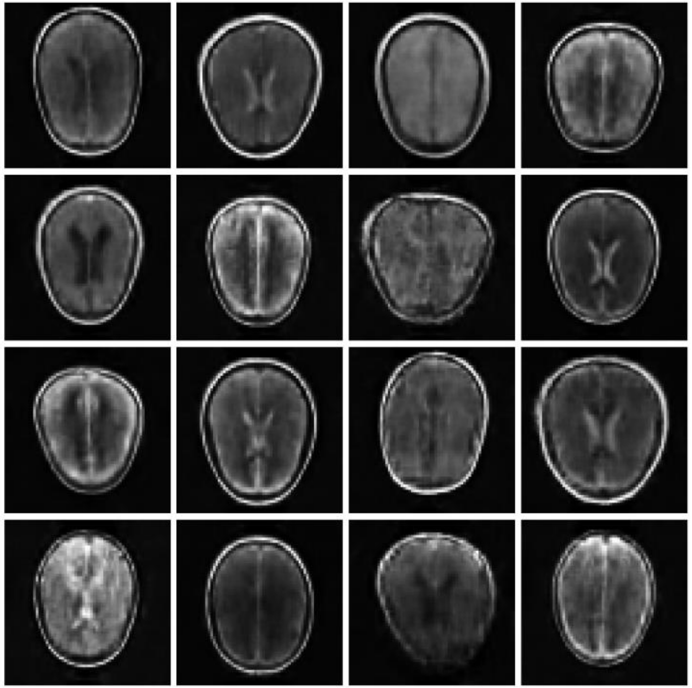
$$y|x,\sigma^2 \sim N(Ax,\sigma^2 I)$$

$$\pi(x) = ?$$

Aim: Reduction of measurement time (sparse observations in K-space)

Application: qualitative MRI

Generative model trained on clinical images: fastmri.org ~ 7000 brain MRIs



Randomly generated from VAE

Summary

- Generative models have a natural interpretation as prior in inverse problems
- Tools from machine learning can be applied
- Variable space prior models may lead to consistent Bayes estimators
- Latent space modelling may lead to inconsistent Bayes estimators

Physikalisch-Technische Bundesanstalt
Braunschweig und Berlin
Abbestraße 2–12
10587 Berlin
Manuel Marschall
Telefon: 030 3481-7948
E-Mail: manuel.marschall@ptb.de
www.ptb.de

Comparison

Latent space approach	Variable space approach
non-linear optimization required	computational infesible integral for every evaluation of prior
manifold restricted → inconsistency*	Not restricted → asympotically consistent
low-dimensional (~10-100)	high-dimensional ($\sim 28^2 - 256^2$)
sampling from prior is easy and fast	sampling from prior is easy and fast

* If x not in image of g

Make the variable approach tractable

Idea: linearize the generator map

$$g(z) = g(z_0) + J_{z_0}(z - z_0), \;\; \Gamma(z) = \Gamma(z_0)$$

$$\pi(x) = \int_{\mathcal{Z}} \pi(x|z)\pi(z)\mathrm{d}z \propto \int_{\mathcal{Z}} |\Gamma(z)|^{-1/2} \exp\left(-\frac{(x - g(z))^T \Gamma(z)^{-1}(x - g(z))}{2} - \frac{z^T z}{2}\right) \mathrm{d}z$$

$$\pi(x) \approx \pi_L(x) = N\left(x | g(z_0) - J_{z_0} z_0, \Gamma(z_0) + J_{z_0} J_{z_0}^T\right)$$

Comparison again

Latent space approach	Variable space approach (Laplace)
non-linear optimization required	Analytic inversion possible
manifold restricted → inconsistency*	Not restricted → asympotically consistent
low-dimensional (~10)	high-dimensional ($\sim 28^2$)
sampling from prior is easy and fast	Sampling from prior is difficult

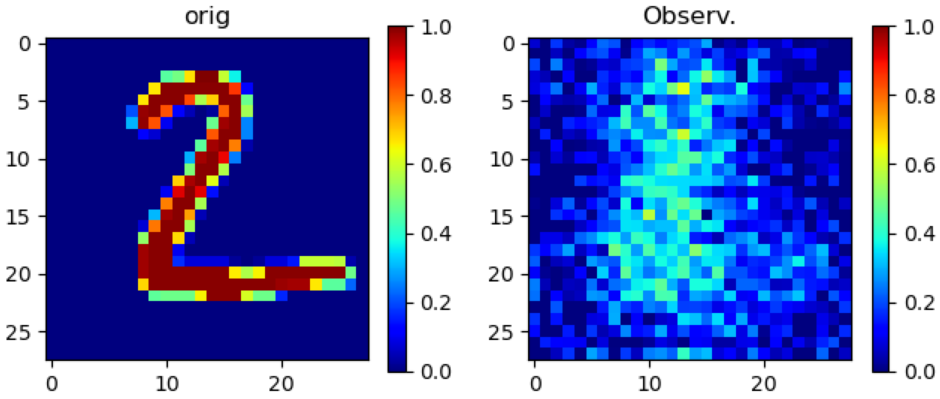
* If x not in image of g

Comparison again

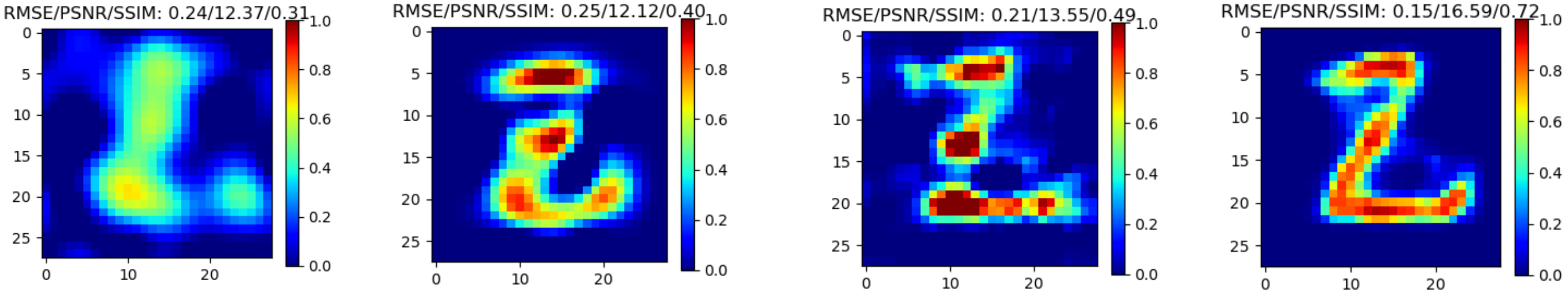
Latent space approach	Variable space approach (Laplace)
non-linear optimization required	Analytic inversion possible
manifold restricted → inconsistency*	Not restricted → asympotically consistent
low-dimensional (~10)	high-dimensional $(\sim 28^2)$
sampling from prior is easy and fast	Sampling from prior is difficult

* If x not in image of g

Numerical results



$$\sigma = 0.1$$



$N(0, \lambda^{-2})$ $\quad N(mean, cov)$ $\quad Laplace$ $\quad Latent$